PRECALCULUS TOPIC II
TRANSFORMING SUBSETS OF $\mathbb{R}^2$

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1. Transformations of Subsets of $\mathbb{R}^2$

Let S be a subset of $\mathbb{R}^2$; that is, S is a set of ordered pairs of real numbers, and we denote this fact as $S \subset \mathbb{R}$. This set may be viewed as a picture in the Cartesian plane. It is possible to transform this set by shifting, stretching, and reflecting, by making adjustments to the coordinates of the points in the set S.

For example, to shift the set to the right by 3 units, we need to add 3 to the x-coordinate of every point in S. We indicate this transformation by the notation $x \mapsto x + 3$, read "x maps to $x + 3$".

The reader should check that all of the transformations listed below have the claimed effect.

Shift right h units	$x \mapsto x + h$
Shift left h units	$x \mapsto x - h$
Shift up k units	$y \mapsto y + k$
Shift down k units	$y \mapsto y - k$
Stretch horizontally by a factor of a	$x \mapsto ax$
Shrink horizontally by a factor of a	$x \mapsto \frac{x}{a}$
Stretch vertically by a factor of b	$y \mapsto by$
Shrink vertically by a factor of b	$y \mapsto \frac{y}{b}$
Reflect across the y-axis	$x \mapsto -x$
Reflect across the x-axis	$y \mapsto -y$

`Transformations of Subsets of` $\mathbb{R}^2$

Keep in mind that the chart above is appropriate only for modifying coordinates of points in a set. More commonly, we would like to modify an equation which produces the set. The rules for this are different.

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2. Transformations of Equations

The *locus* of an equation in the variables x and y is the subset of $\mathbb{R}^2$ consisting of points (x, y) which satisfy the equation. By subtracting the right-hand side of the given equation, we obtain an equivalent equation of the form $F(x, y) = 0$, where $F(x, y)$ is an expression involving x and y.

Consider the case of transforming a set $S \subset \mathbb{R}^2$, where S is the locus of such an equation $F(x, y) = 0$. For each of the transformations of the set S listed above, there is a corresponding transformation of the equation $F(x, y) = 0$ so that the locus of the new equation is the transformed set S.

It turns out that the operations on the equation $F(x, y) = 0$ are exactly the opposite of the operations listed above. This is caused by the fact that to test whether or not a given point is in the transformed set, we need to move that point back to the place where the original equation is valid, then test this moved point with the original equation.

With a bit of experimentation, one convinces oneself that this is true. Let's look at a couple of examples.

Example 1. Let S be the locus of the equation $x^2 + y^2 - 1 = 0$; we know that S is a circle of radius one centered at the origin. If we add 3 to the x-coordinate and 4 to the y-coordinate of every point in S, we will translate this set right 3 and up 4, so that the center of the new circle is at $(3, 4)$. We already know that the equation of this new circle is $(x - 3)^2 + (y - 4)^2 = 1$.

Example 2. Again, let S be the locus of the equation $x^2 + y^2 = 1$. We can stretch S by a factor of 7 horizontally and 5 vertically by multiplying the x-coordinates by 7 and the y-coordinates by 5 for every point in X. The transformed set is an ellipse, whose equation in $(\frac{x}{7})^2 + (\frac{y}{5})^2 = 1$; one can check that the x-intercepts of this equation are ± 7, and the y-intercepts of this equation are ± 5.

Shift right h units	$x \mapsto x - h$
Shift left h units	$x \mapsto x + h$
Shift up k units	$y \mapsto y - k$
Shift down k units	$y \mapsto y + k$
Stretch horizontally by a factor of a	$x \mapsto \frac{x}{a}$
Shrink horizontally by a factor of a	$x \mapsto ax$
Stretch vertically by a factor of b	$y \mapsto \frac{y}{b}$
Shrink vertically by a factor of b	$y \mapsto by$
Reflect across the y-axis	$x \mapsto -x$
Reflect across the x-axis	$y \mapsto -y$

`Transformations of Equations` $F(x, y) = 0$

Again, it is important to keep in mind that the chart above is appropriate exactly for the various operations on equations in the variables x and y. More commonly for us, we wish to understand modified functions, and again, this case is slightly different.

3. Transformations of Functions

Let $D \subset \mathbb{R}$ and let $f : D \to \mathbb{R}$, that is, f is a function defined on a set D, which is a subset of the real numbers $\mathbb{R}$, and the function f takes real values. In this case we consider the equation $y = f(x)$; the locus of this equation is the graph of the function. Typically, $f(x)$ is given as some expression in the variable x.

In order to understand how to modify the expression which defines the function f in order to transform the locus, we modify the equation $y = f(x)$ according to the rules above.

In the case of the x variable, the rules for transforming the graph stay the same. However, for the y variable, we solve the modified equation for y to obtain a new function:

- Shift up: $y \mapsto y - k$ in the equation $y = f(x)$. This equation becomes $y - k = f(x)$, which is equivalent to $y = f(x) + k$.
- Stretch vertically: $y \mapsto \frac{y}{b}$ in the equation $y = f(x)$. This equation becomes $\frac{y}{b} = f(x)$, which is equivalent to $y = bf(x)$.
- Reflect vertically: $y \mapsto -y$ in the equation $y = f(x)$. This equation becomes $-y = f(x)$, which is equivalent to $y = -f(x)$.

In summary,

- $f(x) + k$ describes $f(x)$ shifted up by k units.
- $bf(x)$ describes $f(x)$ stretched vertically by a factor of b.
- $-f(x)$ describes $f(x)$ reflected across the x-axis.

Finally, we produce a chart of the transformations of a function which produce transformations of its graph. We produced the other charts primarily so that we can better understand this one; we will often use transformations of functions to understand the graphs of given functions.

Shift right h units	$f(x-h)$
Shift left h units	$f(x+h)$
Shift up k units	$f(x)+k$
Shift down k units	$f(x)-k$
Stretch horizontally by a factor of a	$f(\frac{x}{a})$
Shrink horizontally by a factor of a	$f(ax)$
Stretch vertically by a factor of b	$bf(x)$
Shrink vertically by a factor of b	$\frac{f(x)}{b}$
Reflect across the y-axis	$f(-x)$
Reflect across the x-axis	$-f(x)$

Transformations of Functions $f(x)$

Exercise 1. Let $S = \{(1,2),(2,4),(2,6)\}$. Compute and graph sets obtained from the following transformations of S.

(a) none
(b) shift left 7 and up 3;
(c) shrink horizontally by a factor of 2 and stretch vertically by a factor of 3;
(d) reflect across the x-axis, the y-axis, and both.

Exercise 2. Consider the equation $x^2 + y^2 = 4$. Find and graph the equations obtained by the following transformations of this equation.

(a) none;
(b) shift right 3 and left 4;
(c) shift right 3 and left 4, then stretch horizontally by 2 and vertically by 3;
(d) shift right 3 and left 4, then stretch horizontally by 2 and vertically by 3, then rotate around the origin.

Exercise 3. Consider the function $f(x) = 2x - 4$. Find and graph the functions obtained by the following transformations of this function.

(a) none;
(b) shift right 4;
(c) shift down 3;
(d) stretch horizontally by a factor of 2;
(e) shrink vertically by a factor of 2;
(d) shift left 2 and down 4, then stretch vertically by a factor of 2, then reflect across the y-axis.

Exercise 4. Consider the function $f(x) = x^2$. Find and graph the functions obtained by the following transformations of this function.

(a) none;
(b) shift right 4;
(c) shift down 3;
(d) stretch horizontally by a factor of 2;
(e) shrink vertically by a factor of 2;
(d) shift left 2 and down 4, then stretch vertically by a factor of 2, then reflect across the y-axis.

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